Math 451: Introduction to General Topology
Lecture 8

Criterion for completeness. For a metric space (X,d), TFAE:

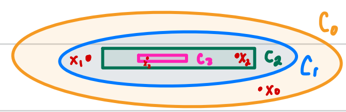
(1) (X,d) is complete.

(2) Every $\subseteq$-decreasing sequence (C_n) of closed nonempty subsets of X of *vanishing diameter*, i.e.
$$\lim_{n\to\infty} \operatorname{diam}(C_n) = 0,$$
has a nonempty intersection: $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$.

(3) Every $\subseteq$-decreasing sequence (B_n) of closed balls in X of vanishing diameter has a nonempty intersection: $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$.

Proof. (2) $\Rightarrow$ (3). Trivial because closed balls are closed sets.

(1) $\Rightarrow$ (2). Suppose (X,d) is complete and let (C_n) be a decreasing sequence of closed sets of vanishing diameter. Use AC to get a sequence (x_n) with $x_n \in C_n$ for each $n \in \mathbb{N}$. Then for each n, the tail $\{x_n, x_{n+1}, \dots\} \subseteq C_n$ because $C_n \supseteq C_{n+1} \supseteq C_{n+2} \supseteq \dots$ Hence



$$\operatorname{diam}\{x_n, x_{n+1}, \dots\} \le \operatorname{diam}(C_n) \to 0 \text{ as } n\to\infty,$$
so (x_n) is Cauchy, hence converges to some $x \in X$. But for each $n \in \mathbb{N}$, $\{x_n, x_{n+1}, \dots\} \subseteq C_n$ and converges to x, so $x \in C_n$ because C_n is closed. Thus, $x \in \bigcap_{n\in\mathbb{N}} C_n$.

(2) $\Rightarrow$ (1). Assume (2) and let $(x_n) \subseteq X$ be Cauchy, i.e. $\operatorname{diam}\{x_n, x_{n+1}, \dots\} \to 0$ as $n \to \infty$. Then $C_n := \overline{\{x_n, x_{n+1}, \dots\}}$ are decreasing closed nonempty sets of vanishing diameter because, as shown last time, $\operatorname{diam}(C_n) = \operatorname{diam}\{x_n, x_{n+1}, \dots\} \to 0$ as $n\to\infty$. Thus, $\exists\, x \in \bigcap_{n\in\mathbb{N}} C_n$. But then $d(x, x_n) \le \operatorname{diam}(C_n) \to 0$ as $n\to\infty$, so $\lim_{n\to\infty} x_n = x$.

(3) $\Rightarrow$ (1). Let (x_n) be a Cauchy sequence.

Acceleration trick. Since a Cauchy sequence converges if some subsequence of it converges, we may move to any subsequence and convergence for that. We choose the following subsequence: let n_0 be large enough so that $\operatorname{diam}\{x_{n_0}, x_{n_0+1}, \dots\} < 1$. Suppose $n_0 < n_1 < \dots < n_k$ are choosen, and take as $n_{k+1} > n_k$ a large enough number such that $\operatorname{diam}\{x_{n_{k+1}}, x_{n_{k+1}+1}, \dots\} < 2^{-(k+1)}$. Then $\operatorname{diam}\{x_{n_k}, x_{n_{k+1}}, x_{n_{k+2}}, \dots\} < 2^{-k}$. Moving to this subsequence, we may assume WLOG that the original sequence satisfied $\operatorname{diam}\{x_n, x_{n+1}, x_{n+2}, \dots\} < 2^{-n}$ to begin with, so we don't carry the double-indices with us.

Now assume $\operatorname{diam}\{x_n, x_{n+1}, x_{n+2}, \dots\} < 2^{-n}$ for all $n \in \mathbb{N}$ and take

$$B_n := \bar{B}_{2\cdot 2^{-n}}(x_n).$$

Then $B_n \supseteq B_{n+1}$ because $d(x_n, x_{n+1}) < 2^{-n}$ so if $y \in B_{n+1} = \bar{B}_{2\cdot 2^{-(n+1)}}(x_{n+1})$, then $d(y, x_{n+1}) \le 2 \cdot 2^{-(n+1)} = 2^{-n}$, hence

$$d(x_n, y) \le d(x_n, x_{n+1}) + d(x_{n+1}, y) < 2^{-n} + 2^{-n} = 2 \cdot 2^{-n},$$

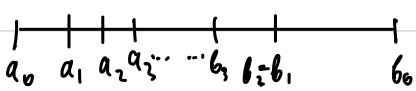
so $y \in \bar{B}_{2\cdot 2^{-n}}(x_n) = B_n$. Finally, since $\operatorname{diam}(B_n) \le 2 \cdot 2 \cdot 2^{-n} \to 0$ as $n \to \infty$, $\exists\, x \in \bigcap_{n\in\mathbb{N}} B_n$. Then $d(x, x_n) \le \operatorname{diam}(B_n) \to 0$ as $n \to \infty$, so $\lim_{n\to\infty} x_n = x$. QED

Caution. The vanishing diameter condition is absolutely necessary, contrary to one's intuition.
For example, take the 0/1-metric d on $X := \mathbb{N}$. Then this is a complete metric space as discussed last time. However, $C_n := \{n, n+1, n+2, \dots\}$ is a closed set (every subset of $\mathbb{N}$ is closed because every subset is open since every singleton $\{n\} = B_{1/2}(n)$ is open). $\operatorname{Diam}(C_n) = 1$ for all $n \in \mathbb{N}$, so $\operatorname{diam}(C_n) \not\to 0$ and $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$.
Another example shows the failure of (3) with balls, and is left as HW.

Examples of complete spaces.

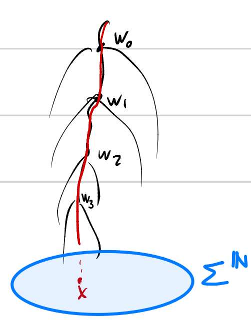
(a) $\mathbb{R}$ with the usual metric is complete.

Proof. Let (B_n) be a decreasing sequence of closed balls in $\mathbb{R}$. Thus, $B_n = [a_n, b_n]$ for some reals $a_n \le b_n$. Then $a_0 \le a_1 \le a_2 \le \dots \le \dots b_n \le b_{n-1} \le \dots \le b_0$, so (a_n) is bounded above (by say b_0) and (b_n) is bounded below (by say a_0), hence $a := \sup_n a_n$ and $b := \inf_n b_n$ exists.



But then for each $n \in \mathbb{N}$, $a_n \le a \le b_n$ and $a_n \le b \le b_n$, so $a_n \le a \le b \le b_n$, hence $\emptyset \ne [a, b] \subseteq \bigcap_{n\in\mathbb{N}} [a_n, b_n]$. $\square$

(b) For each nonempty ctbl Σ, $\Sigma^{\mathbb{N}}$ is complete. HW



Prop. Let (X,d) be a complete metric space. Any subset $Y \subseteq X$ is closed $\iff$ (Y,d) is complete.

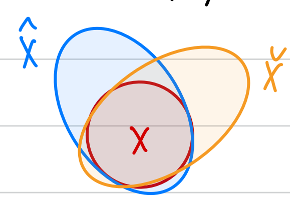
Proof. $\Rightarrow$. Let $(y_n) \subseteq Y$ be a Cauchy sequence. Since X is complete, (y_n) has a limit $x \in X$. But Y is closed so $x \in Y$ hence (y_n) converges in (Y,d) to $x \in Y$.

$\Leftarrow$. Let $(y_n) \subseteq Y$ converging to a point $x \in X$. In particular, (y_n) is Cauchy, so by the completeness of Y, it must have a limit $y \in Y$. But then in X, $y_n \to x$ and $y_n \to y$, so by the uniqueness of limit, $x = y \in Y$. $\boxed{\text{QED}}$

Def. A **completion** of a metric space (X,d) is a complete metric space $(\hat{X},\hat{d})$ such that $\hat{X} \supseteq X$, $\hat{d}|_X = d$, and the closure of X in $(\hat{X},\hat{d})$ is all of $\hat{X}$.

Theorem. Every metric space admits a completion which is unique up to isometric isomorphism; more precisely, if $(\hat{X},\hat{d})$ and $(\check{X},\check{d})$ are two completions of (X,d) then there is a bijective isometry $f: \hat{X} \to \check{X}$ such that $f|_X = id_X$.

Proof of uniqueness. Suppose $(\hat{X},\hat{d})$ and $(\check{X},\check{d})$ are two completions, hence $X \subseteq \hat{X}$, $X \subseteq \check{X}$ and $\overline{X}^{\hat{d}} = \hat{X}$ and $\overline{X}^{\check{d}} = \check{X}$. Define $f: \hat{X} \to \check{X}$ as follows: for each $\hat{x} \in \hat{X}$, take a sequence $(x_n) \subseteq X$ converging to $\hat{x}$ and set $f(\hat{x}) :=$ the limit of (x_n) inside $\check{X}$. This limit exists since (x_n) is d-Cauchy because (x_n) converges to $\hat{x}$ in $(\hat{X},\hat{d})$ and $\check{X}$ is complete. We need to show that f is well-defined, i.e. if $(x_n') \to \hat{x}$ in $\hat{X}$, then still the limit of (x_n') in $\check{X}$ is the same as the limit of (x_n) in $\check{X}$. But since both (x_n) and (x_n') converge to $\hat{x}$ in $(\hat{X},\hat{d})$, we have $d(x_n,x_n') \to 0$ as $n \to 0$, so in $\check{X}$, their limits must coincide.



It remains to verify that f is as desired.

(a) For each $x \in X$, $f(x) = x$ since we can take $x_n := x$.

(b) f is bijective because we can construct f^{-1} exactly the same way as f, but with $\wedge$ and $\vee$ swapped.

(c) f is an isometry. Indeed, let $\hat{x},\hat{y} \in \hat{X}$, choose $(x_n) \to \hat{x}$, $(y_n) \to \hat{y}$ in $\hat{X}$, so $f(\hat{x}) = \lim_{n\to\infty} x_n$ in $\check{X}$, $f(\hat{y}) = \lim_{n\to\infty} y_n$ in $\check{X}$. But $\hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} \hat{d}(x_n,y_n) =$

$$\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} \check{d}(x_n,y_n) = d(\check{x},\check{y}) = d\big(f(\hat{x}),f(\hat{y})\big).$$ $\boxed{\text{QED}}$

The standard proof of the existence of a completion requires taking a quotient by an equivalence relation. Before we start this proof, let's quickly review equivalence relations and quotients.

Equivalence relations and quotients.

Let X be a set. A binary relation R on X is just a subset of X^2. Instead of writing $(x,y)\in R$, we often write xRy to emphasize that R is a relation between x and y. For example, $<$ is a binary relation on $\mathbb{N}$, but we never write $(2,7)\in <$, and instead we write $2<7$.

A binary relation R on X is called an equivalence relation if it is

(i) reflexive: xRx for all $x\in X$;

(ii) symmetric: $xRy \Rightarrow yRx$ for all $x,y\in X$;

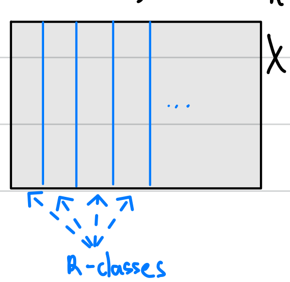
(iii) transitive: $(xRy$ and $yRz) \Rightarrow xRy$ for all $x,y,z\in X$.

For each $x\in X$, call the set
$$[x]_R := \{y\in X : xRy\}$$
the R-class (or R-equivalence class) of x. The main statement is:

Prop. Any two R-classes $[x]_R$ and $[y]_R$ are either equal or disjoint.

Proof. Suppose R-classes $[x]_R$ and $[y]_R$ are not disjoint, $\exists z\in [x]_R\cap[y]_R$, and we show that $[x]_R=[y]_R$. By definition, $z\in[x]_R$ means xRz and $z\in[y]_R$ means yRz. By symmetry, zRy, so by transitivity, xRy since xRz and zRy. This implies that $[y]_R\subseteq[x]_R$ because if $u\in[y]_R$, then yRu hence xRu (i.e. $u\in[x]_R$) by transitivity, since xRy. By symmetry, we also have yRx, so by the same argument, $[x]_R\subseteq[y]_R$, hence $[x]_R=[y]_R$. $\boxed{\text{QED}}$

Denote by X/R the set of R-classes and call it the quotient of X by R. By reflexivity, $x\in[x]_R$ for all $x\in X$. This and the last proposition together imply that X/R is a partition of X:
$$X = \bigsqcup_{C\in X/R} C,$$
where $\sqcup$ indicates that the union is disjoint.



Conversely, given a partition $\mathcal{P}$ of X, i.e. $X = \bigsqcup_{P \in \mathcal{P}} P$, define a binary relation R on X by setting $x R y$:<=> x and y belong to the same P in $\mathcal{P}$. This is easily verified to be an equivalence relation whose classes are exactly the cells P of the partition $\mathcal{P}$.

Thus, equivalence relations on X are in one-to-one correspondence with partitions of X.

Lastly, letting R be an equivalence relation on X, we define the quotient map $\pi : X \to X/R$ by

$$x \mapsto [x]_R.$$

Example (Rational numbers). Assume we have defined $\mathbb{N}$ and $\mathbb{Z}$. What is $\mathbb{Q}$? Well, we write $\frac{n}{m}$ for a rational number, where n, m are integers and $m \neq 0$, but then all of a sudden $\frac{n}{m} = \frac{3n}{3m}$, how? Well, $\frac{n}{m}$ is not the same as a pair (n, m); indeed, $(n,m) \neq (3n, 3m)$. Thus, $\mathbb{Q} \neq \{(n,m) : n, m \in \mathbb{Z} \text{ and } m \neq 0\} =: X$. To get $\mathbb{Q}$ from X we need to "identify" pairs like $(1,2)$, $(3,6)$, and $(-7,-14)$. Formally, this "identification" is just a quotient by the equivalence relation R on X defined by

$$(n,m) \, R \, (k,\ell) \; :\Leftrightarrow \; n\ell = km.$$

One easily checks that R is an equivalence relation and we denote the quotient X/R by $\mathbb{Q}$. We also denote the R-class of (n,m) by

$$\frac{n}{m} := [(n,m)]_R,$$

so of course, $\frac{1}{2} := [(1,2)]_R = [(-7,-14)]_R =: \frac{-7}{-14}$. Also, the map $\mathbb{Z} \to \mathbb{Q}$ by $n \mapsto \frac{n}{1}$ is an injection and we simply identify $\mathbb{Z}$ with its image in $\mathbb{Q}$, so we consider $\mathbb{Z}$ as a subset of $\mathbb{Q}$.